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Section 15

Convergence of martingales. Fundamental Wald's identity.

We finally get to our main result about the convergence of martingales and submartingales.

Theorem 34 *Let* $(X_n, \mathcal{B}_n)_{-\infty<n<\infty}$ *be a submartingale.*

1. *The limit* $X_{-\infty} = \lim_{n\to-\infty} X_n$ *exists a.s. and* $\mathbb{E}X^+_{-\infty} < +\infty$. *If* $\mathcal{B}_{-\infty} = \bigcap_n \mathcal{B}_n$ *then* $(X_n, \mathcal{B}_n)_{-\infty\leq n<\infty}$ *is a submartingale.*
2. *If* $\sup_n \mathbb{E}X^+_n < \infty$, *then* $X_{+\infty} = \lim_{n\to\infty} X_n$ *exists a.s. and* $\mathbb{E}X^+_{+\infty} < \infty$.
3. *If* $(\max(X_n, a))_{-\infty<n<\infty}$ *is uniformly integrable for any* $a \in \mathbb{R}$ *then* $(X_n, \mathcal{B}_n)_{-\infty<n\leq+\infty}$ *is a submartingale where* $X_{+\infty}$ *is from part 2 and* $\mathcal{B}_{+\infty} = \sigma\Big(\bigcup_n \mathcal{B}_n\Big)$.

Remark. The first statement says that a *reverse* submartingale always converges (because it is right-closed by definition). The second statement says that under integrability conditions a submartingale converges. The last statement means that under stronger uniform intergrability conditions the submartingale not only converges but the limit right-closes the submartingale.

Proof. Let us note that X_n converges as $n \to \infty$ $(\infty = \pm\infty)$ if and only if

$$\limsup X_n = \liminf X_n.$$

X_n diverges if and only if the following event occurs

$$\Big\{\limsup X_n > \liminf X_n\Big\} = \bigcup_{a<b} \Big\{\limsup X_n \geq b > a \geq \liminf X_n\Big\},$$

where the union is taken over all rational numbers $a < b$. In other words, $\mathbb{P}(X_n \text{ diverges}) > 0$ iff there exist rational $a < b$ such that

$$\mathbb{P}\Big(\limsup X_n \geq b > a \geq \liminf X_n\Big) > 0.$$

If we recall that $\nu(a, b, n)$ denotes the number of upcrossings of $[a, b]$, this is equivalent to

$$\mathbb{P}\Big(\lim_{n\to\infty} \nu(a, b, n) = \infty\Big) > 0.$$

Proof of 1. Let us define a new sequence

$$Y_1 = X_{-n}, Y_2 = X_{-n+1}, \ldots, Y_n = X_{-1}.$$

By Doob's inequality,

$$\mathbb{E}\nu_Y(a,b,n) \le \frac{\mathbb{E}(Y_n - a)^+}{b-a} = \frac{\mathbb{E}(X_{-1}-a)^+}{b-a} < \infty.$$

Since $0 \le \nu(a,b,n) \uparrow \nu(a,b)$, by monotone convergence theorem,

$$\mathbb{E}\nu(a,b) = \lim_{n\to\infty} \mathbb{E}\nu(a,b,n) < \infty.$$

Therefore, $\mathbb{P}(\nu(a,b)=\infty)=0$ which means that the limit

$$X_{-\infty} = \lim_{n\to-\infty} X_n$$

exists. By Fatou's lemma and submartingale property

$$\mathbb{E}X_{-\infty}^+ \le \liminf_{n\to-\infty} \mathbb{E}X_{-n}^+ \le \mathbb{E}X_{-1}^+ < \infty.$$

It remains to show that $(X_n, \mathcal{B}_n)_{-\infty\le n<\infty}$ is a submartingale. First of all, $X_{-\infty} = \lim_{n\to-\infty} X_n$ is measurable on $\mathcal{B}_m$ for all m and, therefore, measurable on $\mathcal{B}_{-\infty} = \cap\mathcal{B}_m$. Let us take a set $A \in \cap\mathcal{B}_n$. We would like to show that for any m

$$\mathbb{E}X_{-\infty}\mathrm{I}_A \le \mathbb{E}X_m\mathrm{I}_A.$$

Since $(X_n, \mathcal{B}_n)_{-\infty<n\le 0}$ is a right-closed submartingale, by Lemma 29, part 2,

$$\Big(\max(X_n,a), \mathcal{B}_n\Big)_{-\infty<n\le 0}$$

is uniformly integrable for any $a \in \mathbb{R}$. Since

$$\max(X_{-\infty},a) = \lim_{n\to-\infty} \max(X_n,a),$$

by Lemma 30,

$$\mathbb{E}\max(X_{-\infty},a)\mathrm{I}_A = \lim_{n\to-\infty} \mathbb{E}\max(X_n,a)\mathrm{I}_A \le \mathbb{E}\max(X_m,a)\mathrm{I}_A$$

for any m. Letting $a\to-\infty$, by monotone convergence theorem, $\mathbb{E}X_{-\infty}\mathrm{I}_A \le \mathbb{E}X_m\mathrm{I}_A$.

Proof of 2. If $\nu(a,b,n)$ is a number of upcrossings of $[a,b]$ by $X_1,\ldots,X_n$ then by Doob's inequality

$$(b-a)\mathbb{E}\nu(a,b,n) \le \mathbb{E}(X_n-a)^+ \le |a| + \mathbb{E}X_n^+ \le K < \infty.$$

Therefore, $\mathbb{E}\nu(a,b) < \infty$ for $\nu(a,b) = \lim_{n\to\infty}\nu(a,b,n)$ and, as above, $X_{+\infty} = \lim_{n\to+\infty} X_n$ exists. Since all X_n are measurable on $\mathcal{B}_{+\infty}$ so is $X_{+\infty}$. Finally, by Fatou's lemma

$$\mathbb{E}X_{+\infty}^+ \le \liminf_{n\to\infty} \mathbb{E}X_n^+ < \infty.$$

Notice that another condition, $\sup_n \mathbb{E}|X_n| < \infty$, would similarly imply

$$\mathbb{E}|X_{+\infty}| \le \liminf_{n\to\infty} \mathbb{E}|X_n| < \infty.$$

Proof of 3. By 2, the limit $X_{+\infty}$ exists. We want to show that for any set $A \in \mathcal{B}_m$

$$\mathbb{E}X_m\mathrm{I}_A \le \mathbb{E}X_{+\infty}\mathrm{I}_A.$$

Let us denote $a \vee b = \max(a,b)$. Since $(X_n \vee a)$ is uniformly integrable and $X_n \vee a \to X_{+\infty} \vee a$, by Lemma 30,

$$\mathbb{E}(X_{+\infty}\vee a)\mathrm{I}_A = \lim_{n\to\infty} \mathbb{E}(X_n \vee a)\mathrm{I}_A.$$

Since a function $x \to x \vee a$ is convex and increasing, $(X_n \vee a)$ is also a submartingale and, thus, for $n > m$,

$$\mathbb{E}(X_m \vee a)\mathrm{I}_A \le \mathbb{E}(X_n\vee a)\mathrm{I}_A.$$

Therefore, $\mathbb{E}(X_m\vee a)\mathrm{I}_A \le \mathbb{E}(X_{+\infty}\vee a)\mathrm{I}_A$ and letting $a\to-\infty$, by monotone convergence theorem, we get that $\mathbb{E}X_m\mathrm{I}_A \le \mathbb{E}X_{+\infty}\mathrm{I}_A$.

□

As a corollary we get the following.

Corollary 2 *Martingale $(X_n, \mathcal{B}_n)$ is right-closable iff it is uniformly integrable.*

To prove this, apply case 3 above to (X_n) and $(-X_n)$ which are both submartingales.

Theorem 35 *(Levy's convergence) Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and X be a real-valued random variable on it. Given a sequence of σ-algebras*

$$\mathcal{B}_1 \subseteq \ldots \subseteq \mathcal{B}_n \subseteq \ldots \subseteq \mathcal{B}_{+\infty} \subseteq \mathcal{B}$$

where $\mathcal{B}_{+\infty} = \sigma\Big(\bigcup_{1\le n<\infty} \mathcal{B}_n\Big)$, we have $X_n = \mathbb{E}(X|\mathcal{B}_n) \to \mathbb{E}(X|\mathcal{B}_{+\infty})$ a.s.

Proof. $(X_n, \mathcal{B}_n)_{1\le n<\infty}$ is a right-closable martingale since $X_n = \mathbb{E}(X|\mathcal{B}_n)$. Therefore, it is uniformly integrable and by previous theorem the limit $X_{+\infty} = \lim_{n\to\infty} \mathbb{E}(X|\mathcal{B}_n)$ exists. It remains to show that

$$X_{+\infty} = \mathbb{E}(X|\mathcal{B}_{+\infty}).$$

For a set A in algebra $\bigcup \mathcal{B}_n$, $A \in \mathcal{B}_m$ for some m and by Lemma 30

$$\mathbb{E}X_{+\infty}\mathrm{I}_A = \lim_{n\to\infty} \mathbb{E}X_n\mathrm{I}_A = \mathbb{E}X_m\mathrm{I}_A = \mathbb{E}(\mathbb{E}(X|\mathcal{B}_m)\mathrm{I}_A) = \mathbb{E}X\mathrm{I}_A.$$

By uniqueness in the Caratheodory extension theorem, $\mathbb{E}X_{+\infty}\mathrm{I}_A = \mathbb{E}X\mathrm{I}_A$ for all $A \in \sigma\Big(\bigcup_n \mathcal{B}_n\Big)$ which means that $X_{+\infty} = \mathbb{E}(X|\mathcal{B}_{+\infty})$.

□

Exercise. *(Kolmogorov's 0-1 law)* Consider arbitrary random variables $(X_i)_{i\ge1}$ and consider σ-algebras $\mathcal{B}_n = \sigma(X_1, \ldots, X_n)$ and $\mathcal{B}_\infty = \sigma((X_l)_{l\ge1})$. A set $A \in \mathcal{B}_\infty$ is a *tail event* if it is independent of $\mathcal{B}_n$ for any n. If we consider conditional expectations $\mathbb{E}(I_A|\mathcal{B}_n)$ then using Levy's convergence theorem and the fact that $\mathbb{E}(I_A|\mathcal{B}_n)$ and I_A are independent one can prove that $\mathbb{P}(A) = 0$ or 1. This gives a martingale proof of improved Kolmogorov's 0-1 law.

Examples.

1. *(Strong law of large numbers)* Let $(X_n)_{n\ge1}$ be i.i.d. Let

$$\mathcal{B}_{-n} = \sigma(S_n, S_{n+1}, \ldots),\ \ S_n = X_1 + \cdots + X_n.$$

We showed before that

$$Z_{-n} = \frac{S_n}{n} = \mathbb{E}(X_1|\mathcal{B}_{-n}),$$

i.e. $(Z_{-n}, \mathcal{B}_{-n})$ is a reverse martingale and, therefore, the limit $Y = \lim_{n\to+\infty} Z_{-n}$ exists. Y is measurable on σ-algebra $\mathcal{B}_{-\infty} = \bigcap_n \mathcal{B}_{-n}$. Since each event in $\mathcal{B}_{-\infty}$ is symmetric, i.e. invariant under permutations of X_n's, by Savage-Hewitt 0 - 1 law, the probability of each event is 0 or 1, i.e. $\mathcal{B}_{-\infty}$ consists of $\emptyset$ and Ω up to sets of measure zero. Therefore, Y is a constant a.s. and since $(Z_n)_{-\infty\le n\le -1}$ is also a martingale, $\mathbb{E}Y = \mathbb{E}X_1$. Therefore, $S_n/n \to \mathbb{E}X_1$ a.s.

2. *(Kolmogorov's law of large numbers)* Consider a sequence $(X_n)_{n\ge1}$ such that

$$\mathbb{E}(X_{n+1}|X_1, \ldots, X_n) = 0.$$

We do not assume independence. If a sequence (b_n) is such that

$$b_n < b_{n+1},\ \ \lim_{n\to\infty} b_n = \infty \ \ \text{and} \ \ \sum_{n=1}^{\infty} \frac{\mathbb{E}X_n^2}{b_n^2} < \infty$$

then $S_n/n \to 0$ a.s. Indeed, $Y_n = \sum_{k\le n}(X_k/b_k)$ is a martingale and (Y_n) is uniformly integrable since

$$\mathbb{E}|Y_n|\mathrm{I}(Y_n| > M) \le \frac{1}{M}\mathbb{E}|Y_n|^2 \le \frac{1}{M}\sum_{n=1}^{\infty} \frac{\mathbb{E}X_k^2}{b_k^2} \to 0$$

as $M \to \infty$. Therefore, the limit $Y = \lim Y_n$ exists and it is an easy exercise to show that $S_n/b_n \to 0$ (Kronecker's lemma).

3. *(Polya' urn scheme)* Let us recall the Polya urn scheme from Section 5. Let us consider a sequence

$$Y_n = \frac{\#(\text{blue balls after } n \text{ iterations})}{\#(\text{total after } n \text{ iterations})}.$$

Y_n is a martingale because given that at step n the numbers of blue and red balls are b and r, the expected number of balls at step $n+1$ will be

$$\mathbb{E}(Y_{n+1}|\mathcal{B}_n) = \frac{b}{b+r} \cdot \frac{b+c}{b+r+c} + \frac{r}{b+r} \cdot \frac{b}{b+r+c} = \frac{b}{b+r} = Y_n.$$

Since Y_n is bounded, by martingale convergence theorem, the limit $Y = \lim_{n\to\infty} Y_n$ exists. What is the distribution of Y? Let us consider a sequence

$$X_i = \begin{cases} 1 & \text{blue at step } i \\ 0 & \text{red at step } i \end{cases}$$

and let $S_n = \sum_{i\le n} X_i$. Clearly,

$$Y_n = \frac{b + S_n c}{b+r+nc} \approx \frac{S_n}{n}$$

as $n \to \infty$ and, therefore, $S_n/n \to Y$. The sequence (X_n) is exchangeable and by de Finetti's theorem in Section 5 we showed that

$$\mathbb{P}(S_n = k) = \binom{n}{k} \int_0^1 x^k (1-x)^{n-k} d\beta\Bigl(\frac{b}{c}, \frac{r}{c}\Bigr)(x).$$

For any function $u \in C([0,1])$,

$$\mathbb{E}u\Bigl(\frac{S_n}{n}\Bigr) = \sum_{k=0}^{n} u\Bigl(\frac{k}{n}\Bigr)\binom{n}{k} \int_0^1 x^k(1-x)^{n-k} d\beta\Bigl(\frac{b}{c}, \frac{r}{c}\Bigr)(x) = \int_0^1 B_n(x) d\beta\Bigl(\frac{b}{c}, \frac{r}{c}\Bigr)(x),$$

where $B_n(x)$ is the Bernstein polynomial that approximates $u(x)$ uniformly on $[0,1]$. Therefore,

$$\mathbb{E}u\Bigl(\frac{S_n}{n}\Bigr) \to \int_0^1 u(x) d\beta\Bigl(\frac{b}{c}, \frac{r}{c}\Bigr)(x)$$

which means that

$$\mathcal{L}\Bigl(\frac{S_n}{n}\Bigr) \to \beta\Bigl(\frac{b}{c}, \frac{r}{c}\Bigr) = \mathcal{L}(Y),$$

i.e. the limit Y has Beta distribution $\beta(\frac{b}{c}, \frac{r}{c})$.

$\square$

Optional stopping for martingales revisited. Let τ be a stopping time. We would like to determine when $\mathbb{E}X_\tau = \mathbb{E}X_1$. As we saw above in the case of two stopping times, some kind of integrability assumptions are necessary. In this simpler case, the necessary conditions are clear from the proof.

Lemma 31 *We have*

$$\mathbb{E}X_1 = \lim_{n\to\infty} \mathbb{E}X_\tau \mathrm{I}(\tau \le n) \Longleftrightarrow \lim_{n\to\infty} \mathbb{E}X_n \mathrm{I}(\tau \ge n) = 0.$$

Proof. We can write,

$$\begin{aligned} \mathbb{E}X_\tau \mathrm{I}(\tau \le n) = \sum_{1\le k\le n} \mathbb{E}X_k \mathrm{I}(\tau = k) &= \sum_{1\le k\le n} \Bigl(\mathbb{E}X_k \mathrm{I}(\tau \ge k) - \mathbb{E}X_k \mathrm{I}(\tau \ge k+1)\Bigr) \\ \Bigl\{\text{since } \{\tau \ge k+1\} = \{\tau \le k\}^c \in \mathcal{B}_k\Bigr\} &= \sum_{1\le k\le n} \Bigl(\mathbb{E}X_k \mathrm{I}(\tau \ge k) - \mathbb{E}X_{k+1} \mathrm{I}(\tau \ge k+1)\Bigr) \\ &= \mathbb{E}X_1 - \mathbb{E}X_{n+1}\mathrm{I}(\tau \ge n+1). \end{aligned}$$

Example. Given $0 < p < 1$, consider i.i.d. random variables $(\xi_i)_{i\geq 1}$ such that

$$\mathbb{P}(\xi_i = 1) = p,\ \ \mathbb{P}(\xi_i = -1) = 1 - p$$

and consider a random walk $X_0 = 0, X_{n+1} = X_n + \xi_{n+1}$. Consider two integers $a \leq -1$ and $b \geq 1$ and define a stopping time

$$\tau = \min\{k \geq 1, X_n = a \text{ or } b\}.$$

If $q = 1 - p$ then $Y_n = (q/p)^{X_n}$ is a martingale since

$$\mathbb{E}(Y_{n+1}|\mathcal{B}_n) = p\Big(\frac{q}{p}\Big)^{X_n+1} + q\Big(\frac{q}{p}\Big)^{X_n-1} = \Big(\frac{q}{p}\Big)^{X_n}.$$

It is easy to show that $\mathbb{P}(\tau \geq n) \to 0$ as $n \to \infty$ and using previous lemma,

$$1 = \mathbb{E}Y_0 = \mathbb{E}Y_\tau = \Big(\frac{q}{p}\Big)^b \mathbb{P}(X_\tau = b) + \Big(\frac{q}{p}\Big)^a (1 - \mathbb{P}(X_\tau = b)).$$

If $p \neq 1/2$ we can solve this

$$\mathbb{P}(X_\tau = b) = \frac{(q/p)^a - 1}{(q/p)^a - (q/p)^b}.$$

If $q = p = 1/2$ then X_n is a martingale and

$$0 = b\mathbb{P}(X_\tau = b) + a(1 - \mathbb{P}(X_\tau = b)) \Longrightarrow \mathbb{P}(X_\tau = b) = \frac{a}{a-b}.$$

Exercise. Compute $\mathbb{E}\tau$. *Hint:* for $p \neq 1/2$, use that $X_n - n\mathbb{E}\xi_1$ is a martingale; for $p = 1/2$, use that $X_n^2 - n$ is a martingale.

□

Fundamental Wald's identity. Let $(X_n)_{n\geq 1}$ be an i.i.d. sequence of random variables and suppose that a *Laplace transform* $\varphi(\lambda) = \mathbb{E}e^{\lambda X_1}$ is defined on some nontrivial interval (λ_-, λ_+) containing 0. Since

$$\mathbb{E}e^{\lambda S_n} = (\mathbb{E}e^{\lambda X_1})^n = \varphi(\lambda)^n \Longrightarrow \frac{e^{\lambda S_n}}{\varphi(\lambda)^n} \text{ is a martingale.}$$

Let τ be a stopping time. If

$$\mathbb{E}\frac{e^{\lambda S_n}}{\varphi(\lambda)^n} \mathrm{I}(\tau \geq n) \to 0 \tag{15.0.1}$$

then by the above lemma we get

$$\mathbb{E}\frac{e^{\lambda S_\tau}}{\varphi(\lambda)^\tau} \mathrm{I}(\tau < \infty) = \mathbb{E}\frac{e^{\lambda X_1}}{\varphi(\lambda)} = 1$$

which is called the *fundamental Wald's identity.* In some cases one can use this to obtain Laplace transform of a stopping time and, thus, its distribution.

Example. Let $X_0 = 0, \mathbb{P}(X_i = \pm 1) = 1/2, S_n = \sum_{k\leq n} X_k$. Given integer $z \geq 1$, let

$$\tau = \min\{k : S_k = -z \text{ or } z\}.$$

Since $\varphi(\lambda) = \mathrm{ch}(\lambda) \geq 1$,

$$\mathbb{E}\frac{e^{\lambda S_n}}{\varphi(\lambda)^n} \mathrm{I}(\tau \geq n) \leq \frac{e^{|\lambda| z}}{\mathrm{ch}(\lambda)^n} \mathbb{P}(\tau \geq n)$$

and (15.0.1) is satisfied. Therefore,

$$1 = \mathbb{E}\frac{e^{\lambda S_\tau}}{\varphi(\lambda)^\tau} = e^{\lambda z}\mathbb{E}\mathrm{ch}(\lambda)^{-\tau}\mathrm{I}(S_\tau = z) + e^{-\lambda z}\mathbb{E}\mathrm{ch}(\lambda)^{-\tau}\mathrm{I}(S_\tau = -z) = \frac{e^{\lambda z} + e^{-\lambda z}}{2}\mathbb{E}\mathrm{ch}(\lambda)^{-\tau}$$

by symmetry. Therefore,

$$\mathbb{E}\mathrm{ch}(\lambda)^{-\tau} = \frac{1}{\mathrm{ch}(\lambda z)} \quad \text{and} \quad \mathbb{E}e^{\gamma\tau} = \frac{1}{\mathrm{ch}(\mathrm{ch}^{-1}(e^{-\gamma})z)}$$

by a change of variables $e^{\gamma} = 1/\mathrm{ch}\lambda$.

□

For more general stopping times the condition (15.0.1) might not be easy to check. We will now show another approach that is helpful to verify a fundamental Wald's identity. If $\mathbb{P}$ is the distribution of X_i's, let $\mathbb{P}_\lambda$ be a distribution with Radon-Nikodym derivative w.r.t. $\mathbb{P}$ given by

$$\frac{d\mathbb{P}_\lambda}{d\mathbb{P}} = \frac{e^{\lambda x}}{\varphi(\lambda)}.$$

This is, indeed, a density since

$$\int_{\mathbb{R}} \frac{e^{\lambda x}}{\varphi(x)} d\mathbb{P} = \frac{\varphi(\lambda)}{\varphi(\lambda)} = 1.$$

We will think of (X_n) as defined on the product space $(\mathbb{R}^\infty, \mathcal{B}^\infty, \mathbb{P}^\infty)$. For example, a set

$$\{\tau = n\} \in \sigma(X_1, \ldots, X_n) \subseteq \mathcal{B}^n$$

is a Borel set on $\mathbb{R}^n$. We can write,

$$\begin{aligned} \mathbb{E}\frac{e^{\lambda S_\tau}}{\varphi(\lambda)^\tau}\mathrm{I}(\tau < \infty) = \sum_{n=1}^{\infty} \mathbb{E}\frac{e^{\lambda S_n}}{\varphi(\lambda)^n}\mathrm{I}(\tau = n) &= \sum_{n=1}^{\infty} \int\limits_{\{\tau=n\}} \frac{e^{\lambda(x_1+\cdots+x_n)}}{\varphi(\lambda)^n} d\mathbb{P}(x_1) \ldots d\mathbb{P}(x_n) \\ &= \sum_{n=1}^{\infty} \int\limits_{\{\tau=n\}} d\mathbb{P}_\lambda(x_1) \ldots d\mathbb{P}_\lambda(x_n) = \mathbb{P}_\lambda(\tau < \infty). \end{aligned}$$

This means that we can think of (X_n) as having a distribution $\mathbb{P}_\lambda$ and to prove Wald's identity we need to show that $\mathbb{P}_\lambda(\tau < \infty) = 1$.

Example. *(Crossing a growing boundary)* Suppose that we have a boundary given by a sequence $(f(k))$ that changes with time and a stopping time (crossing time):

$$\tau = \min\{k : S_k \geq f(k)\}.$$

To verify (15.0.1), we need to show that $\mathbb{P}_\lambda(\tau < \infty) = 1$. Under the law $\mathbb{P}_\lambda$, random variables X_i have expectation

$$\mathbb{E}_\lambda X_i = \int x \frac{e^{\lambda x}}{\varphi(\lambda)} d\mathbb{P}(x) = \frac{\varphi'(\lambda)}{\varphi(\lambda)}.$$

By the strong law of large numbers

$$\lim_{n\to\infty} \frac{S_n}{n} = \frac{\varphi'(\lambda)}{\varphi(\lambda)}$$

and, therefore, if the growth of the crossing boundary satisfies

$$\limsup_{n\to\infty} \frac{f(n)}{n} < \frac{\varphi'(\lambda)}{\varphi(\lambda)}$$

then, obviously, $\mathbb{P}_\lambda(\tau < \infty) = 1$.

□